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Max-algebra and pairwise comparison matrices ☆

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Abstract

The max-eigenvector of a symmetrically reciprocal matrix A can be used to construct a transitive matrix that is closest to A in a relative error measure. As an alternative to the Perron eigenvector, the max-eigenvector can be used successfully for ranking in the analytical hierarchy process. When either one measurement is corrected or a new alternative is added, the max-eigenvector gives more consistent rankings. Some properties of the max-eigenvector that are important in this process are discussed, and an $O(n^3)$ procedure to calculate the max-eigenvector is detailed.
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1. Introduction

In decision making, given $n \geqslant 3$ alternatives $A_1, \ldots, A_n$, the number a_{ij} indicates the strength with which alternative A_i dominates alternative A_j with respect to a given criterion. The numbers a_{ij} are usually deduced from an experiment, opinion poll or voting behavior. In this way a pairwise comparison matrix can be constructed. Such an n-by-n positive matrix $A = (a_{ij})$ has $a_{ij}a_{ji} = 1$ for all $i, j = 1, \ldots, n$, thus $a_{ii} = 1$. Matrices having these properties are called *symmetrically reciprocal*

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matrices (SR-matrices) (see, for example [7,9]). These matrices were introduced by Saaty [11] and used in the analytic hierarchy process (AHP) for multicriteria decision making. It is required to deduce positive weights $w_1, \dots, w_n$ attached to the alternatives $A_1, \dots, A_n$ respectively, from the SR-matrix A. In this way the alternatives can be ranked. For the ideal case, $a_{ij} = w_i/w_j$ for $i, j = 1, \dots, n$. Matrix A is then an SR-matrix of rank one, which is called a *transitive* (consistent) matrix. However, in a realistic case a_{ij} is only approximately given by w_i/w_j.

There are several suggestions in the literature for constructing a weight vector $w = (w_1, \dots, w_n)$. Saaty [11] himself proposes taking the Perron vector of A and gives some reasoning for that. In [7] the vector w is chosen in such a way that the matrix with the entries w_i/w_j has minimal distance from A in the Euclidean matrix norm, i.e., that

$$\sum_{i,k=1}^{n} (a_{ik} - w_i/w_k)^2$$

is minimal (see also [12]).

Here we propose a different choice, namely taking the max-eigenvector of A. This concept is explained in Section 2. Besides the formal similarity to the Saaty procedure, it turns out that for an SR-matrix A the max-eigenvector x also solves a useful optimization problem, namely minimizing the *relative error*

$$e(w) = \max_{i,k} \left| \frac{a_{ik} - w_i/w_k}{a_{ik}} \right|. \tag{1}$$

Numerically the max-eigenvector can be determined in an $O(n^3)$ procedure (see, e.g., [5]).

We now give a short overview of this paper. In Section 2, the max-algebra is introduced and some important properties of the max-eigenvalue and max-eigenvector for a nonnegative matrix A are given. In Section 3, a max-eigenvector is shown to be a solution to the optimization problem described above. In Section 4, properties of the max-eigenvector that play a role in the AHP are studied. When one pair of entries of A is changed, monotonicity properties of the eigenvector are given. These have implications in the ranking question. When new alternatives are introduced, mathematically augmenting the SR-matrix A by an additional row and column, in many cases the eigenvector is changed only in a manner that does not affect the ranking. Finally, a MATLAB program to calculate the max-eigenvalue and a max-eigenvector of a positive matrix A is given, followed by some numerical examples to demonstrate the use of the max-eigenvector in rankings.

2. Max-eigenvalues and max-eigenvectors

The max-algebra we consider here is the set R_+ of nonnegative real numbers, where for $a, b \in R_+$ the sum $a \oplus b$ is defined as $\max\{a, b\}$ and the product is defined as the usual product ab. For vectors $x = (x_i)$, $y = (y_i)$ in R_+^n and $c \in R_+$ the vectors

$x \oplus y = (\max\{x_i, y_i\})$ and $cx = (cx_i)$ are defined elementwise. The sum $A \oplus B$ of two matrices is defined analogously.

If $A = (a_{ik})$ is a nonnegative n-by-n matrix then the map

$$x \in R_+^n \Longrightarrow A \otimes x \in R_+^n,$$

where $(A \otimes x)_i = \max_k a_{ik}x_k$, $i = 1, \dots, n$ is linear in the sense given above, namely for all $x, y \in R_+^n$, $c \in R_+$

$$A \otimes (x \oplus y) = (A \otimes x) \oplus (A \otimes y),\ A \otimes (cx) = c(A \otimes x).$$

The max-product $C = (c_{il}) = A \otimes B$ of two n-by-n nonnegative matrices $A = (a_{ik})$ and $B = (b_{kl})$ is defined by $c_{il} = \max_k a_{ik}b_{kl}$, $i, l = 1, \dots, n$. It describes the map by B followed by the map by A. The k-fold power of A using this matrix product is denoted by $A_\otimes^k$. It is shown in [2] and in [1], where the max-plus algebra (which is isomorphic to the max-algebra) is treated, that for irreducible A there is a unique positive number $\mu = \mu(A)$ and a vector $x > 0$ such that

$$A \otimes x = \mu(A)x. \tag{2}$$

Here $\mu(A)$ is called the *max-eigenvalue* of A and x is a *max-eigenvector*. The max-eigenvalue has some important and interesting representations. Firstly, $\mu(A)$ is the maximal geometric cycle-mean of A. Given a cycle of length r described by a sequence of integers $i_1, i_2, \dots i_r$, $i_1 \in \{1, \dots, n\}$, then

$$(a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_r i_1})^{1/r} \leqslant \mu(A),$$

and there is one cycle so that equality is attained.

Also as shown in [6], for irreducible A

$$\mu(A) = \min_{v>0} \max_{i,k} a_{ik} v_k / v_i = \max_{i,k} a_{ik} x_k / x_i, \tag{3}$$

where $x = (x_1, \dots, x_m)^{\mathrm{T}}$ is a max-eigenvector of A as in (2).

Generically the max-eigenvector is, up to a scaling, uniquely defined. In [2] the cases with more than one max-eigenvector are described; see [5] and Section 5 for algorithms to calculate the eigenvectors. We conclude this section by the observation that for an SR-matrix A, all cycle products of length two are 1, hence $\mu(A) \geqslant 1$. If, in addition, all cycle products of length three are 1, i.e., $a_{ik}a_{kl}a_{li} = 1$, equivalently $a_{ik}a_{kl} = a_{il}$ then all cycle products are 1. In this case $\mu(A) = 1$ and it follows that $a_{ij} = u_i/u_j$, $i, j = 1, \dots, n$, where the u_i are given, for example, by the first column of A. Thus A is a transitive matrix, and by the above considerations has max-eigenvalue 1. Trivially, for a transitive matrix $A = (u_i/u_j)$, the max-eigenvector is given by (u_i), which coincides with the Perron vector.

3. Relative errors and the max-eigenvector

We begin with the following inequality that is straightforward to verify.

Lemma 1. *For positive numbers a, b, ϵ the following are equivalent.*

$$\frac{1}{1+\epsilon} \leqslant ab \leqslant 1+\epsilon, \tag{4}$$

$$\left|a-\frac{1}{b}\right| \leqslant \epsilon a \quad \textit{and} \quad \left|\frac{1}{a}-b\right| \leqslant \epsilon\frac{1}{a}. \tag{5}$$

A simple consequence of this Lemma is the following result that shows how the max-eigenvector of an SR-matrix A is used to construct a transitive matrix B that minimizes the distance from A in the relative error measure.

Theorem 2. *Let $A=(a_{ik})>0$ be an n-by-n SR-matrix, $w=(w_1,\ldots,w_n)^{\mathrm{T}}$ a positive n-vector and $B=(b_{ik})$, where $b_{ik}=w_i/w_k$. Let $\epsilon>0$. Then the following statements are equivalent*

$$|a_{ik}-b_{ik}| \leqslant \epsilon a_{ik}, \quad i,k=1,\ldots,n, \tag{6}$$

$$a_{ik}w_k/w_i \leqslant 1+\epsilon, \quad i,k=1,\ldots,n. \tag{7}$$

In particular, choosing $w_i=x_i$, $i=1,\ldots,n$, where $x=(x_1,\ldots,x_n)^{\mathrm{T}}$ is a max-eigenvector of A with max-eigenvalue $\mu(A)$, then $\epsilon=\mu(A)-1$ is minimal, and

$$\min_{w>0} e(w) = e(x) = \max_{i,k}\left|\frac{a_{ik}-x_i/x_k}{a_{ik}}\right| = \mu(A)-1. \tag{8}$$

Proof. Applying Lemma 1 with $a=a_{ik}$, and $b=w_k/w_i$ for all $i \leqslant k$ and using the fact that A is an SR-matrix shows the equivalence of (6) and (7). Note that since A is an SR-matrix the two inequalities of (4) are equivalent, and similarly the two inequalities of (5) are equivalent.

It follows from (1) that for any $w>0$

$$1+e(w) = \max_{i,k} a_{ik}w_k/w_i.$$

By (3) the right hand side is minimized by choosing w as a max-eigenvector of A. Hence this choice also minimizes $e(w)$, proving (8). □

So the problem of minimizing $e(w)$ is equivalent to scaling A with a diagonal matrix $W=\mathrm{diag}(w_1,\ldots,w_n)$, namely

$$A \longrightarrow W^{-1}AW, \quad a_{ik} \longrightarrow a_{ik}w_k/w_i,$$

such that the maximal element of the scaled matrix is minimal. We remark that the positions of the maximal elements in $W^{-1}AW$ also give the positions where a_{ik} is furthest away from w_i/w_k. This is an additional advantage of our approach.

4. The behavior of the max-eigenvector

It is well-known and follows easily from each of the representations of the max-eigenvalue that $\mu(A)$ is a monotonically nondecreasing function of the entries of A.

However, nothing seems to be known about the change of the max-eigenvector as A changes. Knowing its behavior is important in the AHP, in which the ranking of the alternatives A_i is often more important than the actual values of the weights w_i attached to the alternatives by the AHP.

There are two types of changes to the original problem that are considered in the literature. Firstly, some measurement in the pairwise comparison SR-matrix A is corrected, i.e., some a_{ij} is replaced by $a_{ij}t$ (and hence a_{ji} by a_{ji}/t) for $t > 0$. This is studied in Theorems 5 and 6. Secondly, a new alternative is added. This amounts to bordering A by a new row and column. Does this change the weights, in our case the entries of the max-eigenvector, and if so, does it change the ranking? We study this question in Theorem 7. As it turns out the max eigenvector behaves more consistently than the Perron vector.

We first treat the case that exactly one pair of entries is changed, and w.l.o.g. assume that this is the pair $(1, 2)$ and $(2, 1)$. Let $A = (a_{ij}) > 0$ be an n-by-n SR-matrix. For $t > 0$ define the SR-matrix $A(t) = (a_{ij}(t))$ by

$$a_{ij}(t) = \begin{cases} a_{12}t & \text{when } i = 1,\ j = 2, \\ a_{21}/t & \text{when } i = 2,\ j = 1, \\ a_{ij} & \text{otherwise}. \end{cases}$$

and the continuous function $\mu(t) = \mu(A(t))$. We then have the following lemmas that lead to the main results in Theorems 5 and 6.

Lemma 3. *Let $n \geqslant 3$. There exist positive numbers $T_1 \leqslant T_2$ such that $\mu(t)$ is strictly decreasing in $(0, T_1]$, is constant in $[T_1, T_2]$ and strictly increasing in $[T_2, \infty)$. For $t \in (0, T_1)$ and for $t \in (T_2, \infty)$ the max-eigenvector is unique up to scaling. Fixing the eigenvector $x(t)$, e.g., by setting the last entry to 1, the eigenvector is continuous in t. If $I = (T_1, T_2) \neq \phi$, then either the max-eigenvector is unique for all $t \in I$ or the eigenvector is nonunique for all $t \in I$.*

Proof. Each cycle mean of $A(t)$ is either strictly increasing in t (if it contains the term $a_{12}(t)$ and not $a_{21}(t)$), strictly decreasing (if it contains the term $a_{21}(t)$ and not $a_{12}(t)$), or is constant in the other cases. As $\mu(t)$ is the maximum of the cycle means of $A(t)$, the first claim follows.

If $t > T_2$ then all maximal cycles contain the factor $a_{12}(t)$, so they are connected. By a well known result, e.g., in [2], the max-eigenvector is uniquely defined. The case $t < T_1$ is treated similarly. A simple analysis using the boundedness of the vectors $x(t)$ shows continuity in these cases.

For all $t \in I$ the maximal cycles of $A(t)$ are the same, so they are either connected, giving uniqueness, or they are not, in which case the eigenvectors are not unique. □

We remark that in general there can be nonuniqueness of the max-eigenvector in $[T_1, T_2]$. For $n = 3$, it can be shown that $I = \phi$.

Lemma 4. *In the situation of Lemma* 3 *define* $\mu_0 = \min\{\mu(t), t > 0\}$. *Then the following are equivalent.*

(i) $\mu_0 = 1$.
(ii) $I = \phi$.
(iii) *The rank of* $A(t)$ *is* 1 *for some* $t = t_0$.

Proof. We introduce the nonnegative n-by-n matrix $B = (b_{ij})$, where $b_{ij} = a_{ij}$ for all (i, j) except that $b_{12} = b_{21} = 0$. Let $x > 0$ be a max-eigenvector of B associated with the max-eigenvalue $\mu(B)$. Clearly, as $B \leqslant A(t)$ for any t, $\mu(B) \leqslant \mu_0$. We show that $\mu(B) = \mu_0$.

Assume first that $\mu(B) = 1$. Then by scaling $A(t)$ and B with the max-eigenvector x, all entries of $A(t)$ are one except those in positions $(1, 2)$ and $(2, 1)$. For suitable t_0 these are also one, so $A(t_0)$ has rank one and $\mu(t) > 1$ for all other values of t. Hence $t_0 = T_1 = T_2$, $\quad \mu_0 = \mu(B)$, and $I = \phi$. Now (i) implies $\mu(B) = 1$, and thus (i) $\rightarrow$ (ii).

Assume now that $\mu(B) > 1$. Then $A(t) \otimes x$ coincides with $B \otimes x = \mu(B)x$ except in entries 1 and 2. These are given by $\mu(B)x_1 \oplus a_{12}tx_2$ and $\mu(B)x_2 \oplus x_1a_{21}/t$ and are equal to $\mu(B)x_i$ iff

$$1/\mu(B) \leqslant ta_{12}x_2/x_1 \leqslant \mu(B) \tag{9}$$

holds. So also in this case $\mu(B) = \mu_0$, $A(t) \otimes x = \mu(B)x$ holds, and hence in this situation $I \neq \phi$. Thus (ii) $\longrightarrow$ (i). The equivalence of (i) and (iii) is trivial. $\square$

Observe that (9) gives upper and lower bounds on T_1 and T_2 respectively. These are however not sharp. It is possible to improve the bounds. We can show that for all t satisfying

$$\mu_0^{-2} \max_{j \geqslant 3} a_{j2}x_2/x_j \leqslant a_{12}tx_2/x_1 \leqslant \mu_0^2 \left(\max_{j \geqslant 3} a_{j1}x_1/x_j \right)^{-1}, \tag{10}$$

we have $\mu(t) = \mu_0$. The proof is somewhat lengthy and not given in this paper.

For the max-eigenvectors we have the following result.

Theorem 5. *Let* $0 < t_0 < t_1$ *and* $x, y > 0$ *be max-eigenvectors of* $A(t)$, *so that*

$$A(t_0) \otimes x = \mu(t_0)x, \quad A(t_1) \otimes y = \mu(t_1)y.$$

(i) *If* $\mu(t_0) < \mu(t_1)$ *then* $y_1/x_1 > y_i/x_i$ *for* $i \geqslant 2$.
(ii) *If* $\mu(t_0) > \mu(t_1)$ *then* $y_2/x_2 < y_i/x_i$ *for* $i \neq 2$.

Proof. Note that for $m \geqslant 2$, it follows from the definition of $A(t)$ that $a_{mk}(t_1) \leqslant a_{mk}(t_0)$. Now assume the contrary of (i), namely that $\mu(t_0) < \mu(t_1)$ and for some $m \geqslant 2$

$$y_m/x_m = \max_i y_i/x_i.$$

The following chain of inequalities holds:

$$\begin{aligned} 0 < \mu(t_0)y_m/x_m &< \mu(t_1)y_m/x_m \\ &= 1/x_m \max_k a_{mk}(t_1)(y_k/x_k)x_k \\ &\leqslant 1/x_m \max_k a_{mk}(t_0)(y_m/x_m)x_k \\ &= y_m/x_m \max_k a_{mk}(t_0)x_k/x_m \\ &= \mu(t_0)y_m/x_m. \end{aligned} \tag{11}$$

This contradiction proves (i).

The second result follows from (i) by interchanging the roles of $A(t_0)$ and $A(t_1)$ and exchanging rows and columns 1 and 2. □

We notice that the proof in the previous theorem follows closely that of Thm. 2.1 in [3] that treats the standard algebra case. We also notice that in the proof of (i) the fact that only one element in the first row is increased is not used. Also it is only used that the entries in the other rows are not increased, so the two matrices involved need not be SR-matrices.

Remark. We discuss here briefly the case $I = \phi$ (see Lemma 4). Assume that $t_0 = 1$, which can be achieved by a shift, so that $A = A(1)$ is of rank 1. Then we can show that $\mu(t) = \max(t^{1/3}, t^{-1/3})$ and that a max-eigenvector $x(t)$ is given for all $t > 0$ by

$$x(t) = (t^{1/3}, a_{21}t^{-1/3}, a_{31}, \ldots, a_{n1})^{\mathrm{T}}. \tag{12}$$

Hence such a perturbation of a transitive matrix only affects two entries of the weight vector. Rank reversal, i.e., when the order of the first and second entry is reversed, appears exactly at $t = a_{21}^{3/2}$. This very simple behavior should be contrasted with the complicated behavior of the Perron vector in this situation, as discussed, e.g., in [8].

We now consider the more complicated case of equality.

Theorem 6. *Under the conditions and assumptions of Theorem* 5 *assume that* $\mu(t_0) = \mu(t_1)$.

(i) *If there is* $\tau \in (t_0, t_1)$ *such that* $\mu(\tau) < \mu(t_0)$ *then*

$$y_1/x_1 \geqslant y_i/x_i \geqslant y_2/x_2, \quad i = 1, \ldots, n \tag{13}$$

and in addition

$$y_1/x_1 > y_2/x_2. \tag{14}$$

(ii) *If on the other hand* $t_j \in (T_1, T_2)$ *for* $j = 0, 1$ *and at least for one* $t \in \{t_0, t_1\}$ *the max-eigenvector is unique*, *then inequality* (13) *also holds.*

Proof. Consider the first case, in which $t_0 < T_1$ and $t_1 > T_2$. Let $A(\tau) \otimes z = \mu(\tau)z$. Now apply Theorem 5, (ii) to x and z. This yields

$$z_2/x_2 < z_i/x_i, \quad i \neq 2, \tag{15}$$

and applying Theorem 5, (i) to z and y leads to

$$y_1/z_1 > y_i/z_i, \quad i \geqslant 2. \tag{16}$$

Multiplying (15) with $i = 1$ and (16) with $i = 2$ immediately gives (14). Now let $\tau \to t_0$ from above. Then suitably scaled, $z \to x$, and from (16) the left inequalities in (13) hold. Similarly considering $\tau \to t_1$ from below yields the second inequalities in (13).

The proof of case (ii) is very technical and thus placed in Appendix A. □

In the last part of this section we consider the situation that a new alternative is added, equivalently that the SR-matrix A is bordered by a new row and column. Starting from an n-by-n SR-matrix A with max-eigenvalue μ and max-eigenvector x, so that

$$A \otimes x = \mu x, \tag{17}$$

the extended SR-matrix A_e is constructed from A by bordering with a positive vector $u = (u_1, \ldots, u_n)^{\mathrm{T}}$. Specifically,

$$A_e = \begin{pmatrix} A & u \\ u^{-1} & 1 \end{pmatrix}, \tag{18}$$

where $u^{-1} = (u_1^{-1}, \ldots, u_n^{-1})$. We see in the following that for a large set of vectors $u > 0$ the max-eigenvalue is not increased and that the max-eigenvector is just extended by a suitable number.

Theorem 7. *Assume that* $\mu > 1$ *and take* A_e *as in* (18) *with* A *as in* (17). *The following are equivalent*:

(i) $\mu(A) = \mu(A_e) = \mu$, *and there is* $\alpha > 0$ *such that* $(x^{\mathrm{T}}, \alpha)^{\mathrm{T}}$ *is the max-eigenvector of* A_e.
(ii) $\max_k(u_k/x_k)\max_k(x_k/u_k) \leqslant \mu^2$.

Sufficient conditions for u to satisfy (i) *and* (ii) *are*: *there exists* $h > 0$ *such that* $u = A \otimes h$, *or there exists* $h > 0$ *such that* $u = Ah$, *the standard matrix–vector product.*

Proof. If (i) holds, then

$$A \otimes x \oplus \alpha u = \mu x, \quad u^{-1} \otimes x \oplus \alpha = \mu\alpha, \tag{19}$$

and this implies that $\alpha u \leqslant \mu x$ and $u^{-1} \otimes x = \mu\alpha$. This gives $\alpha u_i/x_i \leqslant \mu$, $i = 1, \ldots, n$ and $\max(x_i/u_i) = \mu\alpha$, and (ii) follows.

To show the reverse implication, choose $\alpha = \mu^{-1} \max_i (x_i/u_i)$. Then the second equation in (19) is satisfied. By (ii) the first set of equations (19) also holds, thus (i) holds.

To prove the second part it suffices to consider the case $x = e_n = (1, \ldots, 1)^{\mathrm{T}}$, so that $1/\mu \leqslant a_{ik} \leqslant \mu$ for $i, k = 1, \ldots, n$. So we have $a_{ik} \leqslant \mu^2 a_{jk}$ for all i, j, k, which implies for any $h > 0$ that

$$(A \otimes h)_i \leqslant \mu^2 (A \otimes h)_j \quad \forall i, j \tag{20}$$

and

$$(Ah)_i \leqslant \mu^2 (Ah)_j \quad \forall i, j. \tag{21}$$

Hence (ii) is satisfied in both cases. □

In particular, repeating an alternative, i.e., taking as u any column of A, does not change $\mu(A)$ and changes the max-eigenvector only in a trivial way, e.g., if u is the first column of A, then a max-eigenvector of A_e is $(x_1, \ldots, x_n, x_1)^{\mathrm{T}}$, as one would expect in this model. This is in contrast to the more complicated behaviour of the Perron-eigenvector (see, e.g., [8]).

It seems that these sufficient conditions in Theorem 7 do not cover all possibilities. Here is an interesting example of a vector u satisfying (i) and (ii), but for which we were not able to decide if it can be written as $A \otimes h$ or Ah for a suitable $h > 0$.

Take

$$u_i = a_{i1}^{\alpha_1} \cdots a_{in}^{\alpha_n}, \quad i = 1, \ldots, n, \tag{22}$$

where the exponents α_i are nonnegative and sum up to 1. Writing

$$\frac{u_i}{x_i} = \prod_{k=1}^{n} (a_{ik} x_k / x_i)^{\alpha_k} \left(\prod_{k=1}^{n} x_k^{\alpha_k} \right)^{-1}, \tag{23}$$

and using the fact that $a_{ik} x_k / x_i \in [\mu^{-1}, \mu]$, it follows that (ii) is true. For $\alpha_k = 1/n$, $k = 1, \ldots, n$, this vector appears in the logarithmic least squares method considered in [12].

Usually, extending the SR-matrix A as in (18) strictly increases its max-eigenvalue, $\mu(A_e) > \mu(A)$ if (ii) of Theorem 7 is violated. However, it is possible to find a 4×4 matrix A and an extension by some u violating (ii) but having $\mu(A_e) = \mu(A)$. Of course, in this case, the other part of (i) does not hold.

Remark. The Perron eigenvector of A satisfies the second sufficient condition of Theorem 7. Hence (ii) holds for the Perron vector. This partly explains the observation that for SR-matrices the max-eigenvector and the Perron vector are quite similar and lead to very similar rankings of the alternatives. For general nonnegative matrices this similarity of eigenvectors is not usually observed.

We finally consider the case dual to the situation of Theorem 7, namely that A has rank one, i.e., $\mu(A) = \mu = 1$; thus A is a transitive matrix. Here we determine completely the behavior of A_e for general u in all detail. Such a discussion seems to be too complicated for the Perron vector (see, e.g., [9]).

Theorem 8. *Assume that* $A = (a_{ik}) = (w_i/w_k)$ *and* $w_i > 0, i = 1, \ldots, n$. *Also let* $u = (u_1, \ldots, u_n)^{\mathrm{T}} > 0$ *and* $d_i = u_i/w_i$. *Define* $d_{\max} = \max_i d_i$ *and* $d_{\min} = \min_j d_j$. *If the* $(n+1)$*-by-*$(n+1)$ *matrix* A_e *is given by* (18) *then*

$$\mu_e = \mu(A_e) = (d_{\max}/d_{\min})^{1/3}, \tag{24}$$

while an associated max-eigenvector $x = (x_1, \ldots, x_n, 1)^{\mathrm{T}}$ *is given by*

$$x_i = \begin{cases} w_i d_{\max}^{1/3} d_{\min}^{-2/3} & \textit{if } d_i \leqslant d_{\max}^{2/3} d_{\min}^{1/3} \\ u_i \mu_e^{-1} & \textit{otherwise.} \end{cases} \tag{25}$$

Proof. Define $d^{\mathrm{T}} = (d_1, \ldots, d_n)$. Then scaling A_e by $W = \mathrm{diag}(w_1, \ldots, w_n, 1)$ gives

$$W^{-1} A_e W = \tilde{A}_e = \begin{pmatrix} e_n e_n^{\mathrm{T}} & d \\ d^{-1} & 1 \end{pmatrix}. \tag{26}$$

The eigenequation for $\tilde{A}_e$ in the form $\tilde{A}_e \otimes \tilde{x} = \mu_e \tilde{x}$ where $\tilde{x}^{\mathrm{T}} = (x^{\mathrm{T}}, 1)$ leads to $e_n e_n^{\mathrm{T}} \otimes x \oplus d = \mu_e x$ and $d^{-1} \otimes x \oplus 1 = \mu_e$. As the case $\mu_e = 1$, i.e., that d is a multiple of e_n is trivial, we now consider that $\mu_e > 1$. Then the second eigenequation gives $\mu_e = d^{-1} \otimes x$, while the first leads to

$$x = \mu_e^{-1} e_n e_n^{\mathrm{T}} \otimes x \oplus \mu_e^{-1} d, \tag{27}$$

with the solution (see, e.g., [5, Lemma 3.3])

$$x = (I \oplus \mu_e^{-1} e_n e_n^{\mathrm{T}}) \otimes \mu_e^{-1} d = \mu_e^{-1} d \oplus \mu_e^{-2} (e_n^{\mathrm{T}} \otimes d) e_n. \tag{28}$$

Thus

$$\begin{aligned} \mu_e &= d^{-1} \otimes x \\ &= \mu_e^{-1} (d^{-1} \otimes d) \oplus \mu_e^{-2} (e_n^{\mathrm{T}} \otimes d)(d^{-1} \otimes e_n) \\ &= \max\{\mu_e^{-1}, \mu_e^{-2} d_{\max}/d_{\min}\} \end{aligned} \tag{29}$$

from which (24) follows.

Inserting this in (28) gives

$$x_i = \begin{cases} d_{\max}^{1/3} d_{\min}^{-2/3} & \text{if } d_i \leqslant d_{\max}^{2/3} d_{\min}^{1/3} \\ d_i \mu_e^{-1} & \text{otherwise,} \end{cases} \tag{30}$$

which after scaling back and renaming yields (25). □

5. Calculating the max-eigenvector

One of the advantages of our approach is that the max-eigenvalue and the max-eigenvector can be calculated easily. A simple MATLAB program is given below for any nonnegative irreducible matrix A. Let us briefly explain the procedure, as outlined, e.g., in [5, Section 4].

Firstly the max-eigenvalue of the nonnegative irreducible matrix A is calculated using Karp's formula: With $z_1 = (1, 0, \dots, 0)^{\mathrm{T}}$ and $z_{k+1} = A \otimes z_k, \quad k = 1, \dots, n$ and $z_k = (z_{1k}, \dots, z_{nk})^{\mathrm{T}}$, Theorem. 3.1 of [5] gives

$$\mu(A) = \max_{i=1,\dots,n} \min_{k=1,\dots,n} \left(\frac{z_{i,n+1}}{z_{ik}}\right)^{1/(n+1-k)}. \tag{31}$$

Notice that to handle the zeros in z_1 properly, we actually calculate μ^{-1} first and then invert.

As explained, e.g., in [1, p. 147–148], a max-eigenvector of a matrix A with $\mu(A) = 1$ can be determined in the following way. Calculate the matrix

$$A^+ = A \oplus A^2_\otimes \oplus \dots \oplus A^n_\otimes. \tag{32}$$

Then any column j of A^+ with $(A^+)_{jj} = 1$ is a max-eigenvector of A. The condition $\mu(A) = 1$ is enforced by replacing A by $A/\mu(A)$ and A^+ is calculated by the Floyd-Warshall procedure, as, e.g., described in [5, (4.2)].

The following MATLAB program *maxevec* calculates the max-eigenvalue *mu* and a max-eigenvector *evec* of a given nonnegative irreducible matrix a.

```
%Given square matrix a, its max eigenvalue mu=mu(a) and an
%eigenvector evec is determined. mu is calculated by Karp's
%formula. The eigenvector evec is calculated by the
%Floyd-Warshall procedure, as the first column of b^+,
%b=a/mu(a) which has diagonal entry 1.

function[mu,evec]=maxevec(a)[m,n]=size(a);

z=[];

for i=1:n z(1,i)=0; end;

z(1,1)=1;

for i=2:n+1

   w=z(i-1,:);

   w=max(diag(w)*a');

   z=[z;w];
```

```
end

z1=z*diag(w.^-1);

z1 =z1(1:n,:); for i=1:n

  t= z1(i,:);  z1(i,:)= t.^(1/(n+1-i));

end

mu=min(max(z1)); mu =mu^-1;

b=mu^{-1}*a;

for i=1:n

   w=b(:,i); u= b(i,:);

   c=w*u;

   for j=1:n

      for k=1:n b(j,k)=max(c(j,k),b(j,k));

      end; end;

end;

tol=10^-10;

for i=1:n

   if abs(b(i,i)-1) < tol

      evec=b(:,i); return;

      end; end;
```

We now give some numerical examples to illustrate the advantage of using the max-eigenvector (as calculated by the program above) for ranking. We emphasize that these examples illustrate the use of the max-eigenvector approach for finding the priority ranking of the alternatives with respect to a given criterion.

The first example is from [7].

$$A = \begin{bmatrix} 1 & 1/5 & 3 & 3 \\ 5 & 1 & 5 & 3 \\ 1/3 & 1/5 & 1 & 1/5 \\ 1/3 & 1/3 & 5 & 1 \end{bmatrix}.$$

Here $\mu(A) = 1.9680$, and the max-eigenvector scaled so that its entries sum to 1 is

$$x^{\mathrm{T}} = (0.2245,\ 0.5703,\ 0.0580,\ 0.1473).$$

The Perron vector scaled in the same way, which gives the same ranking, is

$$p^{\mathrm{T}} = (0.2247,\ 0.5502,\ 0.0622,\ 0.1629).$$

The relative errors are $e(x) = 0.9680$ and $e(p) = 1.1750$, respectively. Since A has one maximal cycle of length 4 (namely, $a_{14}a_{43}a_{32}a_{21}$), the max-eigenvector of A^{T} is $(x_i^{-1})^{\mathrm{T}}$.

An example in which the max-eigenvector and Perron vector give different rankings is the following matrix, which results from a transitive matrix perturbed in the (1, 2) and (2, 1) entries.

$$A = \begin{bmatrix} 1 & 10/9 & 2/3 & 5/8 \\ 9/10 & 1 & 4/5 & 3/4 \\ 3/2 & 5/4 & 1 & 15/16 \\ 8/5 & 4/3 & 16/15 & 1 \end{bmatrix},$$

giving

$$x^{\mathrm{T}} = (0.2080,\ 0.2061,\ 0.2835,\ 0.3024),\ \text{with } e(x) = 0.1006;$$
$$p^{\mathrm{T}} = (0.2034,\ 0.2113,\ 0.2832,\ 0.3021),\ \text{with } e(p) = 0.1543.$$

As $t = 4/3 > (6/5)^{3/2}$, a rank reversal between the first two alternatives as given by the max-eigenvector has occured (see Remark after Theorem 5), while this has not yet occured in the Perron vector.

Our third and final example is taken from [12], and illustrates the repetition of an alternative as described after Theorem 7.

$$A = \begin{bmatrix} 1 & 1/6 & 1/3 & 1/8 & 5 \\ 6 & 1 & 2 & 1 & 8 \\ 3 & 1/2 & 1 & 1/2 & 5 \\ 8 & 1 & 2 & 1 & 5 \\ 1/5 & 1/8 & 1/5 & 1/5 & 1 \end{bmatrix}.$$

Here the max-eigenvector is

$$x^{\mathrm{T}} = (1,\ 3,\ 1.5,\ 4,\ 0.4),\ \text{with } e(x) = 1,$$

while a Perron vector is given by

$$p^{\mathrm{T}} = (0.0810,\ 0.3459,\ 0.1801,\ 0.3548,\ 0.0382),\ \text{with } e(p) = 1.357.$$

Both vectors lead to the same ranking of the alternatives, but p exhibits a near tie (between alternatives 2 and 4), while x does not. Extending A to a 6-by-6 matrix by repeating column 2 and completing to an SR-matrix, the new max-eigenvector is just

$$x^{\mathrm{T}} = (1,\ 3,\ 1.5,\ 4,\ 0.4,\ 3),$$

while the new Perron vector (scaled to have the same first component as before) is

$$p^{\mathrm{T}} = (0.0810,\ 0.3588,\ 0.1859,\ 0.3652,\ 0.0407,\ 0.3588),$$

in which the tie is tighter than before. However, if the extension is done by repeating column 1 (instead of column 2), then the tie between alternatives 2 and 4 is gone.

6. Concluding remarks

In Sections 3 and 4 we propose using the max-eigenvector of the SR-matrix A as a mean of attaching weights to the alternatives considered in the AHP.

The advantages of this approach over others presented in the literature are now summarized. The max-eigenvector gives an approximation with minimum relative error (Theorem 2). When one pair of elements in a transitive matrix is changed (due to some measurement error), then the magnitude of the change giving rank reversal is explicitly given. For an SR-matrix, monotonicity properties of the max-eigenvectors of these changed matrices are found (Theorem 5 and 6). When an alternative is repeated, no rank reversal occurs, and in many cases the introduction of a new alternative just extends the max-eigenvector (Theorem 7). Finally, the max-eigenvector can be easily computed in an explicit fashion (Section 5).

We also want to mention some open problems. There are cases in which the max-eigenvector is not unique (even up to scaling). This happens if the critical matrix A^{C} is reducible. What does this mean for our method? Closely related to this question is the following consideration. There are more vectors satisfying the minimal relative error requirement (8), namely all vectors in C_A, where

$$C_A = \{v > 0 : A \otimes v \leqslant \mu(A)v\}.$$

Are there better choices than the max-eigenvector to pick for an element in C_A? In our first example of Section 5, $C_A = \{kx : k > 0\}$ where x is the max-eigenvector of A; thus there is no other choice for this example.

Appendix A. Proof of (13) in case (ii) of Theorem 6

Let $T_1 < t_0 < t_1 < T_2$. It suffices to show (13) for $t_1 - t_0$ small. Then by multiplying the inequalities it follows that (13) holds for all $t_0, t_1 \in I$. By scaling $A(t_0)$ it can be assumed that $x = e_n$. In the following we set $A = A(t_0)$, $\bar{A} = A(t_1)$. Introduce the matrix $\tilde{A} = (\tilde{a}_{ij})$, where

$$\tilde{a}_{ij} = \begin{cases} \mu_0 & \text{when } a_{ij} = \mu_0 \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

with $\mu_0 = \min\{\mu(t), t > 0\} = \mu(t_0) = \mu(t_1)$. The critical matrix A^{C}, see, e.g., [4,5], is a principal submatrix of $\tilde{A}$. Each row of $\tilde{A}$ contains a nonzero element, and the principal submatrix A^{C} is irreducible, as it is assumed that the max-eigenvector $x(t)$ for all $t \in I$ is unique (see Lemma 3).

If $a_{12} < \mu_0$, then for small $t_1 - t_0$ it follows that $\overline{A} \otimes e_n = \mu_0 e_n$, so (13) holds with equality. If $a_{12} = \mu_0$, then a_{12} is not on a maximal cycle, as otherwise the max-eigenvalue is strictly increasing. So $\bar{a}_{12} = \mu_0 + \epsilon > \mu_0$.

By a suitable permutation similarity, we can now assume that $\tilde{A}$ is in reducible normal form (see, e.g., [10]). So $\tilde{A}$ is lower block triangular with diagonal blocks that are either irreducible or zero 1-by-1 matrices. Here A^{C} is the only irreducible principal submatrix, hence

$$\tilde{A} = \begin{pmatrix} A^{\mathrm{C}} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \tag{34}$$

where $\tilde{A}_{22}$ is strictly lower triangular, and the critical element a_{12} is now (after the permutation) in some row, say $s+1$, of $\tilde{A}$ lying in the lower part $(\tilde{A}_{21}, \tilde{A}_{22})$. We assume that the same permutation has been applied to A and $\overline{A}$. Define A_s to be the submatrix of A formed from the first s rows and columns of A. Then

$$A = \begin{pmatrix} A_s & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{35}$$

Subdivide $\overline{A}$ accordingly into blocks, then due to the way that s has been chosen, $\overline{A}_s = A_s$, $\overline{A}_{22} = A_{22}$.

Observe that $A_s \otimes e_s = \mu_0 e_s$ and

$$(z_{s+1}, \ldots, z_n)^{\mathrm{T}} = \overline{A}_{21} \otimes e_s \tag{36}$$

satisfies $z_{s+1} = \mu_0 + \epsilon$, $z_i \leqslant \mu_0$, for $i \geqslant s+2$, as only row $s+1$ is increased, when A is replaced by $\overline{A}$.

The solution of

$$\overline{A}_{21} \otimes e_s \oplus A_{22} \otimes u = \mu_0 u \tag{37}$$

is given by (see, e.g., Lemma 3.3 of [5])

$$u = (1/\mu_0 A_{22})^* \otimes 1/\mu_0 \bar{A}_{21} \otimes e_s = 1/\mu_0 (1/\mu_0 A_{22})^* \otimes z. \tag{38}$$

Here

$$(1/\mu_0 A_{22})^* = I \oplus \mu_0^{-1} A_{22} \oplus \cdots \oplus (\mu_0^{-1} A_{22})_{\otimes}^{n-s-1}, \tag{39}$$

which has diagonal 1 and all other entries <1. Hence the first entry of u is given by $(\mu_0 + \epsilon)/\mu_0$, while the other entries of u are strictly smaller. Also as all entries of $\overline{A}_{12}$ are strictly less than μ_0, it follows that $\overline{A}_{12} \otimes u \leqslant \mu_0 e_s$ for small ϵ. Thus

$$A_s \otimes e_s \oplus \overline{A}_{12} \otimes u = \mu_0 e_s, \tag{40}$$

which shows together with (37) that $y = (e_s^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$ is the max-eigenvector of $\overline{A}$.

Obviously the maximal element of y is in position $s+1$, which after back permutation becomes position 1. So the first inequalities of (13) are proved. As in the proof of Theorem 5 the second set of inequalities follow by interchanging rows and columns 1 and 2 and the roles of t_0 and t_1.

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